

**PUTTING STRING/STRING DUALITY
TO THE TEST**M. J. Duff[†] and R. Minasian*Center for Theoretical Physics
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After simultaneous compactification of spacetime and worldvolume on $K3$, the $D = 10$ heterotic fivebrane with gauge group $SO(32)$ behaves like a $D = 6$ heterotic string with gauge group $SO(28) \times SU(2)$, but with Kac–Moody levels different from those of the fundamental string. Thus the string/fivebrane duality conjecture in $D = 10$ gets replaced by a string/string duality conjecture in $D = 6$. Since $D = 6$ strings are better understood than $D = 10$ fivebranes, this provides a more reliable laboratory in which to test the conjecture. According to string/string duality, the Green–Schwarz factorization of the $D = 6$ spacetime anomaly polynomial I_8 into $X_4 \tilde{X}_4$ means that just as X_4 is the σ -model anomaly polynomial of the fundamental string worldsheet so \tilde{X}_4 should be the corresponding polynomial of the dual string worldsheet. To test this idea we perform a classical dual string calculation of \tilde{X}_4 and find agreement with the quantum fundamental string result. This also provides an *a posteriori* justification for assumptions made in a previous paper on string/fivebrane duality. Finally we speculate on the relevance of string/string duality to the vacuum degeneracy problem.

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1 Introduction

In D spacetime dimensions an extended object of worldvolume dimension d is dual, in the sense of Poincare duality, to another extended object of dimension \tilde{d} given by

$$\tilde{d} = D - d - 2 \quad (1.1)$$

the most familiar example being $D = 4$, where an electric monopole ($d = 1$) is dual to magnetic monopole ($\tilde{d} = 1$). More recently, attention has focussed on $D = 10$, where a superstring ($d = 2$) is dual to a superfivebrane ($\tilde{d} = 6$) [1]–[55]. In this paper, however, we shall focus on $D = 6$ where a string ($d = 2$) is dual¹ to another string ($\tilde{d} = 2$) [8, 45].

There is now a good deal of circumstantial evidence to suggest that the string and the fivebrane theories may actually be different mathematical formulations of the same physics with the strongly coupled string corresponding to a weakly coupled fivebrane, and vice-versa. We shall refer to this idea as the string/fivebrane duality conjecture. In a certain sense, it is a generalization of the electric/magnetic duality conjecture of [56]. In previous papers [7, 8, 9, 20], it was pointed out that the Green–Schwarz [57] factorization of the spacetime anomaly polynomial I_{12} into $X_4\tilde{X}_8$ provides a non-trivial check on this string/fivebrane duality [1, 2]. Just as X_4 is the σ -model anomaly polynomial of the $d = 2$ string worldsheet, so \tilde{X}_8 should be the corresponding polynomial of the $\tilde{d} = 6$ fivebrane worldvolume. To test this idea, a classical fivebrane calculation of \tilde{X}_8 was performed [22] and found to be in agreement with the string one-loop result, thus supporting the conjecture. Although the fivebrane calculation of the *Yang–Mills* contribution to X_8 is relatively straightforward, however, an educated guess had to be made for the *gravitational* contributions; a guess which has recently been questioned in [43, 44].

In this paper we repeat the test for string/string duality in $D = 6$. Now the factorization of the spacetime anomaly polynomial I_8 into $X_4\tilde{X}_4$ means that \tilde{X}_4 should be the σ -model anomaly polynomial of the $\tilde{d} = 2$ dual string worldsheet. Since $D = 6$ strings are better understood than $D = 10$ fivebranes, this provides a more reliable theoretical laboratory in which to test the conjecture. In particular, the gravitational contributions to the σ -model anomaly for a string are well-known, thus avoiding the guesswork [22] that had to be made for those of the fivebrane. We perform a classical dual string calculation of \tilde{X}_4 and once again find agreement with the fundamental string one-loop result².

The chiral, anomaly-free, fundamental string we shall consider will be the heterotic string obtained by spacetime compactification on $K3$ of the $SO(32)$ heterotic string in $D = 10$. The

¹In fact a string can be dual to another string even in $D < 6$ if the fivebrane wraps around 4 of the compactified directions [2, 42].

²Since the result of [22] is sometimes misunderstood, we stress that two *independent* calculations of \tilde{X}_8 were being compared. The first was the fundamental string one-loop calculation of the spacetime anomaly [57]. The second was the fundamental fivebrane tree-level calculation of the worldvolume anomaly [22]. This latter result is obtained from the fundamental superfivebrane σ -model action of [58] augmented by the internal $SO(32)$ degrees of freedom [19, 20]. Being tree-level, such a calculation is possible in spite of our ignorance of how to quantize the fivebrane. This is, of course, quite different from calculating the string spacetime polynomial [62, 63] from the dual supergravity theory [61, 7] obtained by dualizing the usual string effective action [59, 60, 74], which is guaranteed by construction to yield the same \tilde{X}_8 . Similarly, in the present paper, we will compare two *independent* calculations of \tilde{X}_4 .

resulting $D = 6$ string has gauge group $SO(28) \times SU(2)$ with Kac–Moody levels $k_{SO(28)} = 1$ and $k_{SU(2)} = 1$. The dual string is obtained from the fivebrane by simultaneous compactification of spacetime and worldvolume on $K3$. As discussed in section 3, this string is also chiral and has the same gauge group but with different Kac–Moody levels: $\tilde{k}_{SO(28)} = 4$ and $\tilde{k}_{SU(2)} = -20$ (the minus sign corresponds to the $SU(2)$ worldsheet fermions having the opposite chirality). The calculation of \tilde{X}_4 can be done either directly with the $D = 6$ dual string or else by starting with the $D = 10$ fivebrane and using

$$\tilde{X}_4 = \int_{K3} \tilde{X}_8 \quad (1.2)$$

Both ways yield the same result, and this provides an *a posteriori* justification for the gravitational contributions to \tilde{X}_8 assumed in [22].

2 Review of string/string duality

We begin by recalling some facts about duality in $D = 6$ [45]. There are two formulations of $D = 6$ supergravity, both with a 3-form field strength³ The bosonic part of the usual action takes the form

$$I_6 = \frac{1}{2\kappa^2} \int d^6x \sqrt{-g} e^{-2\Phi} [R + 4(\partial\Phi)^2 - \frac{1}{2 \cdot 3!} H_3^2 + \dots] \quad (2.1)$$

To within Chern–Simons corrections to be discussed below, H_3 is the curl of a 2-form B_2

$$H_3 = dB_2 + \dots \quad (2.2)$$

The metric $g_{\mu\nu} (\mu = 0, \dots, 5)$ is related to the canonical Einstein metric $g^c_{\mu\nu}$ by

$$g_{\mu\nu} = e^\Phi g^c_{\mu\nu} \quad (2.3)$$

where Φ the $D = 6$ dilaton.⁴ Similarly, the dual supergravity action is given by

$$\tilde{I}_6 = \frac{1}{2\kappa^2} \int d^6x \sqrt{-\tilde{g}} e^{2\Phi} [\tilde{R} + 4(\partial\Phi)^2 - \frac{1}{2 \cdot 3!} \tilde{H}_3^2 + \dots] \quad (2.4)$$

To within Chern–Simons corrections to be discussed below, \tilde{H}_3 is also the curl of a 2-form \tilde{B}_2

$$\tilde{H}_3 = d\tilde{B}_2 + \dots \quad (2.5)$$

The metric \tilde{g}_{MN} is related to the canonical Einstein metric by

$$\tilde{g}_{\mu\nu} = e^{-\Phi} g^c_{\mu\nu} \quad (2.6)$$

³There is also a *self-dual* supergravity in $D=6$ and an associated self-dual string [45] for which $H_3 = *H_3$. While interesting in its own right, this is not the subject of the present paper.

⁴Our use of the symbol Φ conforms with the notation of section 6 of [45] where $8\Phi = (D-2)\alpha\phi$ and where $\alpha = \sqrt{2}$ for $D = 6$.

The two supergravities are related by Poincare duality:

$$\tilde{H}_3 = e^{-2\Phi} * H_3 \quad (2.7)$$

where $*$ denoted the Hodge dual. (Since this equation is conformally invariant, it is not necessary to specify which metric is chosen in forming the dual.) This ensures that the roles of field equations and Bianchi identities in the one version of supergravity are interchanged in the other. As field theories, each supergravity seems equally as good. In particular, provided we couple them to $SO(28) \times SU(2)$ super Yang-Mills, then both are anomaly-free.⁵ Since the usual version corresponds to the field theory limit of the heterotic string, it was natural to conjecture [8, 45] that other version corresponds to the field theory limit of a “dual string”.

The bosonic action of the fundamental $D = 6$ heterotic string is given by

$$S_2 = \frac{1}{2\pi\alpha'} \int_{M_2} d^2\xi \left(-\frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i X^\mu \partial_j X^\nu g_{\mu\nu} - \frac{1}{2} \epsilon^{ij} \partial_i X^\mu \partial_j X^\nu B_{\mu\nu} + \dots \right) \quad (2.8)$$

where ξ^i ($i = 1, 2$) are the worldsheet coordinates, γ_{ij} is the worldsheet metric and $(2\pi\alpha')^{-1}$ is the string tension T . The metric and 2-form appearing in S_2 are the same as those in I_6 . This means that under the rescalings with constant parameter λ :

$$\begin{aligned} g_{\mu\nu} &\rightarrow \lambda^2 g_{\mu\nu} \\ B_{\mu\nu} &\rightarrow \lambda^2 B_{\mu\nu} \\ e^{2\Phi} &\rightarrow \lambda^2 e^{2\Phi} \\ \gamma_{ij} &\rightarrow \lambda^2 \gamma_{ij} \end{aligned} \quad (2.9)$$

both actions scale in the same way

$$\begin{aligned} I_6 &\rightarrow \lambda^2 I_6 \\ S_2 &\rightarrow \lambda^2 S_2 \end{aligned} \quad (2.10)$$

We therefore assume that the bosonic action of the dual $D = 6$ heterotic string given by

$$\tilde{S}_2 = \frac{1}{2\pi\tilde{\alpha}'} \int_{\tilde{M}_2} d^2\tilde{\xi} \left(-\frac{1}{2} \sqrt{-\tilde{\gamma}} \tilde{\gamma}^{ij} \partial_i X^\mu \partial_j X^\nu \tilde{g}_{\mu\nu} - \frac{1}{2} \epsilon^{ij} \partial_i X^\mu \partial_j X^\nu \tilde{B}_{\mu\nu} + \dots \right) \quad (2.11)$$

where $\tilde{\xi}^i$ ($i = 1, 2$) are the dual worldsheet coordinates, $\tilde{\gamma}_{ij}$ is the dual worldsheet metric and $(2\pi\tilde{\alpha}')^{-1}$ is the dual string tension \tilde{T} . Furthermore we assume that the metric and 2-form appearing in \tilde{S}_2 are the same as those appearing in \tilde{I}_6 . The virtue of these identifications is that under the recalings with constant parameter $\tilde{\lambda}$:

$$\begin{aligned} \tilde{g}_{\mu\nu} &\rightarrow \tilde{\lambda}^2 \tilde{g}_{\mu\nu} \\ \tilde{B}_{\mu\nu} &\rightarrow \tilde{\lambda}^2 \tilde{B}_{\mu\nu} \\ e^{-2\Phi} &\rightarrow \tilde{\lambda}^2 e^{-2\Phi} \\ \tilde{\gamma}_{ij} &\rightarrow \tilde{\lambda}^2 \tilde{\gamma}_{ij} \end{aligned} \quad (2.12)$$

⁵There are other many other anomaly-free groups in $D = 6$ [62, 64] but this is the one we shall focus on in this paper.

both actions again scale in the same way:

$$\begin{aligned}\tilde{I}_6 &\rightarrow \tilde{\lambda}^2 \tilde{I}_6 \\ \tilde{S}_2 &\rightarrow \tilde{\lambda}^2 \tilde{S}_2\end{aligned}\tag{2.13}$$

The duality relation (2.7) is invariant under both rescalings. Since the dilaton enters the dual string equations with the opposite sign to the fundamental string, it was argued in [8, 45] that the strong coupling regime of the string should correspond to the weak coupling regime of the dual string:

$$g_{\text{dual string}} = \langle e^{-\Phi} \rangle = g_{\text{string}}^{-1}\tag{2.14}$$

where g_{string} and $g_{\text{dual string}}$ are the string and dual string loop expansion parameters.

Further evidence for this duality is provided by the complementary discovery that the dual string emerges as a soliton solution of the fundamental string, and vice-versa [45]. The combined supergravity-source action $I_6 + S_2$ admits the singular elementary string solution [65]

$$ds^2 = (1 - a^2/r^2)(-d\tau^2 + d\sigma^2) + (1 - a^2/r^2)^{-1}dr^2 + (1 - a^2/r^2)r^2 d\Omega_3^2\tag{2.15}$$

$$e^{2\Phi} = 1 - a^2/r^2\tag{2.16}$$

$$e^{-2\Phi} * H_3 = 2a^2 \epsilon_3\tag{2.17}$$

where

$$a^2 = \kappa^2 T / \Omega_3\tag{2.18}$$

and Ω_3 is the volume of S^3 . The source-free action I_6 also admits the non-singular solitonic string solution [45]

$$ds^2 = -d\tau^2 + d\sigma^2 + (1 - \tilde{a}^2/r^2)^{-2}dr^2 + r^2 d\Omega_3^2\tag{2.19}$$

$$e^{-2\Phi} = 1 - \tilde{a}^2/r^2\tag{2.20}$$

$$H_3 = 2\tilde{a}^2 \epsilon_3\tag{2.21}$$

whose tension \tilde{T} is given by

$$\tilde{a}^2 = \kappa^2 \tilde{T} / \Omega_3\tag{2.22}$$

The Dirac quantization rule relating the Noether “electric” charge

$$e = \frac{1}{\sqrt{2}\kappa} \int_{S^3} e^{-2\Phi} * H_3\tag{2.23}$$

to the topological “magnetic” charge

$$g = \frac{1}{\sqrt{2}\kappa} \int_{S^3} H_3\tag{2.24}$$

translates into a quantization condition on the two tensions [5]:

$$2\kappa^2 = n(2\pi)^3 \alpha' \tilde{\alpha}', \quad n = \text{integer}\tag{2.25}$$

Similarly, the dual supergravity-source action $\tilde{I}_6 + \tilde{S}_2$ admits the dual string as the fundamental solution and the fundamental string as the dual solution. When expressed in terms of the dual metric (2.6), however, the former is singular and the latter non-singular. Both the string and dual string soliton solutions break half the supersymmetries, both saturate a Bogomol'nyi bound between the mass and the charge. These solutions are the extreme mass equals charge limit of more general two-parameter black string solutions [13, 45].

Duality mixes up string and dual string loops in the sense that the roles of worldsheet loop expansions and spacetime loop expansions are interchanged [8, 45]. Consequently what is a quantum effect for the string might be a classical effect for the dual string, and vice versa. At higher loop orders, this leads to an infinite number of non-renormalization theorems (including the vanishing of the cosmological term) all of which are consistent with known string calculations to higher orders both in α' (worldsheet loops) and g_{string} (spacetime loops). It is this loop mixing which allows us to test string/string duality, in spite of our ignorance of how to quantize the dual string. If duality is correct, we should be able to reproduce string loop effects from tree-level dual strings! In this paper we shall show in particular how to reproduce the Green–Schwarz spacetime anomaly corrections to the H_3 field equations (a fundamental string one-loop effect) from the Chern–Simons worldsheet anomaly corrections to the \tilde{H}_3 Bianchi identities (a dual string tree-level effect).

3 The fundamental string

The fundamental string we shall consider will be the $D = 6$ heterotic string obtained by compactification of the $D = 10$ heterotic string on $K3$. We choose this because it is essential for our purposes that both the string and the dual string be chiral.

$K3$ is a four-dimensional compact closed simply-connected manifold. It is equipped with a self-dual metric and hence its holonomy group is $SU(2)$. It entered the physics literature as a gravitational instanton [66, 67], its Pontryagin number being

$$p_1 = -\frac{1}{8\pi^2} \int_{K3} \text{tr} R^2 = -48 \quad (3.1)$$

However, it was then invoked in a Kaluza-Klein context in [68, 69] where it was used, in particular, as a way of compactifying $D = 10$ supergravity to $D = 6$. Half the spacetime supersymmetry remains unbroken as a consequence of the $SU(2)$ holonomy, and hence it gives rise to an $N = 1$ supergravity in $D = 6$.

There are four $N = 1, D = 6$ supermultiplets to consider:

<i>Supergravity</i>	$g_{\mu\nu}, \Psi^A_{L\mu}, B_{L\mu\nu}$
<i>Tensor</i>	$B_{R\mu\nu}, \chi^A_R, \phi$
<i>Hypermatter</i>	ψ^a_R, ϕ^α
<i>Yang – Mills</i>	A_μ, λ^A_L

The scalars ϕ^α parametrize a quaternionic Kahler manifold of the form $\mathcal{G}/\mathcal{H} \times Sp(1)$. The index $A = 1, 2$ labels $Sp(1)$ and the index a labels one of the representations of \mathcal{H} . All spinors

are symplectic Majorana–Weyl. The 2-forms $B_{L\mu\nu}$ and $B_{R\mu\nu}$ have 3-form field strengths that are self-dual and anti-self-dual, respectively. Only with the combination of one supergravity multiplet and one tensor multiplet do we have a conventional covariant Lagrangian formulation. In the case of $K3$, the massless sector of the $D = 6$ theory coming from the supergravity multiplet in $D = 10$ consists of this combination plus 20 hypermatter multiplets. The 80 scalars belong to the coset $SO(20, 4)/SO(20) \times SO(3) \times Sp(1)$ [70, 71, 72], this being the moduli space of $K3$. There are no vector multiplets since $K3$ has no isometries and is simply connected.

This argument was generalized to supergravity–Yang–Mills with gauge groups $G_{10} = SO(32)$ or $G_{10} = E_8 \times E_8$ in [73], where it was emphasized that the resulting $N = 1, D = 6$ supergravity would be chiral and anomaly free⁶. Counting the number of $D = 6$ multiplets coming from the Yang–Mills sector is more subtle. As discussed in [79], the values of the background fields, R_0 and F_0 , are not independent. The requirement that H_3 be globally defined leads, on integrating (A.10), to the constraint

$$\int_{M_4} (tr_{SO(32)} F_0^2 - tr_{SO(1,9)} R_0^2) = 0 \quad (3.2)$$

for any closed sub-manifold M_4 of the ten-dimensional spacetime. In this background the effective lower-dimensional theory has a reduced gauge symmetry. If the non-zero fields F_0 span a subgroup $H \subset G_{10}$ then the gauge group in the lower dimension will be given by G such that

$$G_{10} \supset G \times H \quad (3.3)$$

The adjoint representation of G_{10} can be decomposed into a sum of representations

$$adj \ G_{10} = \sum_i (L_i, C_i) \quad (3.4)$$

where L_i and C_i are irreducible representations of G and H respectively. In particular, for $G_{10} = SO(32)$,

$$\sum_i \dim L_i \cdot \dim C_i = \dim G_{10} = 496 \quad (3.5)$$

In the present context we choose M_4 to be $K3$ and the backgrounds R_0 and F_0 to lie only in the four compactified directions. Specifically we “embed the the holonomy group in the gauge group” [80, 74] by taking $F_0=R_0$ to lie in the $SU(2)$ subgroup⁷ of $SO(32)$. Then the Yang–Mills supermultiplets are those of $G = SO(28) \times SU(2)$. Thus

$$\begin{aligned} L_1 &= (378, 1) & C_1 &= 1 \\ L_2 &= (1, 3) & C_3 &= 1 \\ L_3 &= (28, 2) & C_2 &= 2 \\ L_4 &= (1, 1) & C_4 &= 3 \end{aligned}$$

⁶ $K3$ was thus the forerunner of Calabi–Yau compactification [74] from $D = 10$ to $D = 4$. $K3$ has a curious way of cropping up in the superstring literature in a variety of apparently unrelated contexts [68, 69, 71, 72, 73, 75, 76, 77, 78].

⁷Different embeddings were chosen in [73].

The number of left-handed spinor multiplets in the representation L_i of G is given by an index theorem:

$$N_i^{1/2} = \frac{1}{8\pi^2} \int_{M_4} \left[-\frac{1}{2} \text{tr}_{C_i} F_0^2 + \frac{1}{48} \dim C_i \text{tr} R_0^2 \right] \quad (3.6)$$

In the case of $K3$, on using (3.2) and (3.3), this reduces to

$$N_i^{1/2} = (\dim C_i - 12R_i) \quad (3.7)$$

where R_i is the ratio

$$R_i = \frac{\text{tr}_{C_i} F_0^2}{\text{tr} F_0^2} \quad (3.8)$$

So coming from the Yang-Mills sector in $D = 10$ we have

$$\begin{aligned} N_{(378,1)}^{1/2} &= 1 - 0 = 1 \\ N_{(1,3)}^{1/2} &= 1 - 0 = 1 \\ N_{(28,2)}^{1/2} &= 2 - 12 = -10 \\ N_{(1,1)}^{1/2} &= 3 - 48 = -45, \end{aligned} \quad (3.9)$$

the first two being the adjoint representation left-handed gaugino superpartners of the $G = SO(28) \times SU(2)$ gauge fields belonging to the Yang-Mills supermultiplets, and the last two being the right-handed superpartners of ϕ^a in the hypermatter multiplets.

4 Worldsheet and spacetime anomalies

Let us first consider the G Yang-Mills and Lorentz Chern-Simons corrections to H_3 . Let us define $F = dA + A^2$ where the gauge fields $A = A_M dx^M$ are matrices in the adjoint representation of the gauge groups and $R = d\omega + \omega^2$ where the Lorentz connections $\omega = \omega_M dx^M$ are in the vector representation. Let us further define [81]

$$\begin{aligned} I_4 &= \frac{1}{2(2\pi)^2} \left[-\sum_l \frac{k_l}{g_l} \text{Tr} F_l^2 + \text{tr} R^2 \right] \\ d\omega_3 &= I_4 \\ \delta\omega_3 &= d\omega_2^1 \end{aligned} \quad (4.1)$$

where the sum is taken over the gauge groups appearing in $G = G_1 \times G_2 \times \cdots \times G_l$, k_l are the corresponding levels of the Kac-Moody algebra and g_l are the dual Coxeter numbers. Then the action S_2 can be modified so as to be both gauge invariant and Lorentz invariant provided [82]

$$\delta B_2 = \frac{1}{2} \alpha' (2\pi)^2 \omega_2^1 \quad (4.2)$$

and hence the gauge invariant field strength is given by

$$\begin{aligned} H_3 &= dB_2 - \frac{1}{2} \alpha' (2\pi)^2 \omega_3 \\ dH_3 &= -\frac{1}{2} \alpha' (2\pi)^2 I_4 \end{aligned} \quad (4.3)$$

In the case of $K3$ compactification of the $SO(32)$ string, we have

$$SO(32) \supset SO(28) \times SU(2) \quad (4.4)$$

and the fundamental representation decomposes as

$$32 \rightarrow (28, 1) + (1, 2) + (1, 2) \quad (4.5)$$

so the worldsheet fermions carrying the $SO(28) \times SU(2)$ symmetry correspond to Kac–Moody levels $k_{SO(28)} = 1$ and $k_{SU(2)} = 1$. Similarly, the gravitational anomaly is as in $D = 10$ but where R now belongs to $SO(1, 5)$ instead of $SO(1, 9)$. Thus the Yang–Mills and Lorentz Chern–Simons corrections are given by

$$dH_3 = -\frac{\alpha'}{4} \left[-\text{tr} F_{SO(28)}^2 - 2\text{tr} F_{SU(2)}^2 + \text{tr} R^2 \right] \quad (4.6)$$

where we have used $\text{Tr} F_{SO(28)}^2 = 26 \text{tr} F_{SO(28)}^2$, $\text{Tr} F_{SU(2)}^2 = 4 \text{tr} F_{SU(2)}^2$ and $g_{SO(28)} = 26$, $g_{SU(2)} = 2$. This modification to the Bianchi identity is thus a classical string effect (i.e. tree level in the $D = 6$ string loop expansion). This is confirmed by the observation that there is no dilaton dependence and that dH_3 is independent of κ^2 and that the 2π factors cancel out. By supersymmetry, the same combination of Yang–Mills field strengths and curvatures appearing in (4.6) also appears in the $D = 6$ tree level action

$$S^{(0)}_{YM} = \frac{1}{2\kappa^2} \int d^6x \sqrt{-g} e^{-2\Phi} \frac{\alpha'}{8} t^{\mu\nu\rho\sigma} \left[-\text{tr} F_{SO(28)\mu\nu} F_{SO(28)\rho\sigma} - 2\text{tr} F_{SU(2)\mu\nu} F_{SU(2)\rho\sigma} + \text{tr} R_{\mu\nu} R_{\rho\sigma} \right] \quad (4.7)$$

where

$$t^{\mu\nu\rho\sigma} = \frac{1}{2} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \quad (4.8)$$

Next we turn to the Green–Schwarz anomaly cancellation mechanism [57]. The spacetime anomaly polynomial I_8 of this $D = 6$ string has been calculated by Erler [77] who finds, as expected, that it factorizes in the form

$$I_8 = X_4 \tilde{X}_4 \quad (4.9)$$

where

$$X_4 = \frac{1}{2} I_4 = \frac{1}{4(2\pi)^2} \left[-\text{tr} F_{SO(28)}^2 - 2\text{tr} F_{SU(2)}^2 + \text{tr} R^2 \right] \quad (4.10)$$

and

$$\tilde{X}_4 = \frac{1}{4(2\pi)^2} \left[-2\text{tr} F_{SO(28)}^2 + 44\text{tr} F_{SU(2)}^2 - \text{tr} R^2 \right] \quad (4.11)$$

As a consistency check, one notes that factorization requires the absence of a $\text{tr} R^4$ term in I_8 and that this is guaranteed if [73]

$$\sum_{i=1}^4 N_i^{1/2} \dim L_i = -224 \quad (4.12)$$

as may be verified from (3.9).

Defining $\tilde{\omega}_3$ by $\tilde{X}_4 = \frac{1}{2}d\tilde{\omega}_3$, the consistent anomaly is then cancelled by adding to the effective action

$$\Delta\Gamma_2 = -2\pi \int_{M_6} \left(\frac{1}{\alpha'(2\pi)^2} B_2 \tilde{X}_4 + \frac{1}{3} \omega_3 \tilde{\omega}_3 \right) \quad (4.13)$$

and recalling the transformation rule for B_2 given in (4.2). Now with the normalization of the kinetic term for B_2 given in (2.1), the addition of (4.13) modifies the field equation to

$$\begin{aligned} d(e^{-2\Phi} * H_3) &= \frac{2\kappa^2}{\alpha'(2\pi)} \tilde{X}_4 \\ &= \frac{\kappa^2}{2(2\pi)^3\alpha'} \left[-2\text{tr}F_{SO(28)}^2 + 44\text{tr}F_{SU(2)}^2 - \text{tr}R^2 \right] \end{aligned} \quad (4.14)$$

This modification to the field equations is thus a string one-loop effect. This is confirmed by the dilaton dependence on the left hand side and by noting that the right hand side is linear in κ^2 and involves a factor $1/(2\pi)^3$ appropriate to a one-loop Feynman integral in $D = 6$. By supersymmetry, the same combination of Yang-Mills field strengths and gravitational curvatures appearing in (4.14) also appears in the $D = 6$ one loop action

$$\begin{aligned} S^{(1)}_{YM} = \int d^6x \sqrt{-g} \frac{1}{8(2\pi)^3\alpha'} t^{\mu\nu\rho\sigma} \left[-2\text{tr}F_{SO(28)\mu\nu}F_{SO(28)\rho\sigma} \right. \\ \left. + 44\text{tr}F_{SU(2)\mu\nu}F_{SU(2)\rho\sigma} - \text{tr}R_{\mu\nu}R_{\rho\sigma} + \dots \right] \end{aligned} \quad (4.15)$$

After this summary of the $D = 6$ string, we are now in a position to test string/string duality by reproducing (4.14) and (4.15) from the classical dual string.

5 String/string test in $D = 6$

After compactification to $D = 6$, a $D = 10$ fivebrane will appear as a fivebrane, a fourbrane, a threebrane, a membrane or a string according as it wraps around 0, 1, 2, 3 or 4 of the compactified directions. The fivebrane is trivial having no degrees of freedom in $D = 6$. The dual supergravity theory in $D = 6$ obtained by compactifying the dual $D = 10$ supergravity on $K3$, will consist of the same combination of supergravity multiplet and one tensor multiplet as in the fundamental theory but instead of 20 hypermultiplets, there will be 19 linear multiplets of type 1 and one of type 2. In a type 1 linear multiplet [83] one of the four scalars ϕ (an $SU(2)$ singlet) is swapped for a 4-form $b_{\mu\nu\rho\sigma}$; in a type 2 linear multiplet three of the four scalars (an $SU(2)$ triplet) are swapped for three 4-forms. The appearance of these 4-forms means that the dual string alone cannot be responsible for this low-energy limit; the threebrane is also contributing. Only the odd-branes will display *sigma*-model anomalies [19, 22] and we might therefore expect contributions to the $D = 6$ spacetime anomaly polynomial I_8 of the form $X_0\tilde{X}_8$, $X_2\tilde{X}_6$, and $X_4\tilde{X}_4$. However, X_0 is trivially zero and X_2 will involve $\text{tr}R$ which is zero and $\text{tr}F$ which is also zero since our gauge group has no abelian factors. For the purposes of the *sigma*-model anomaly, therefore, we may focus just on the massless states of the dual string worldsheet.

These follow from the compactification of the fivebrane worldvolume \tilde{M}_6 to $\tilde{M}_2 \times K3$. The κ -symmetric, spacetime supersymmetric fivebrane action (in the absence of internal symmetry) has been constructed in [58] using the Green–Schwarz variables x^μ, Θ^α with $\mu = 0, \dots, 9$ and $\alpha = 1, \dots, 16$. Since the scalar d’Alembertian just splits as $\Delta_6 = \Delta_2 + \Delta_{K3}$, the number of massless scalar fields is unchanged under compactification. So the Green–Schwarz variables x^μ will remain the same, except that only six of them will be regarded as spacetime coordinates. Similarly, by spacetime supersymmetry, the Θ^α which transformed as a 16 of $SO(1, 9)$ are now interpreted as a $(4, 4)$ of $SO(1, 5) \times SO(4)$. This counting is exactly the same as that of the fundamental Green–Schwarz string. Hence, the contribution to the gravitational sigma-model anomaly will be the same as that of the fundamental string.

A complete covariant κ -symmetric Green–Schwarz action for the *heterotic* fivebrane (i.e. with the internal symmetry included) is still lacking. If the field theory limit of the heterotic fivebrane is indeed to coincide with the $SO(32)$ anomaly-free supergravity, however, we must include this internal symmetry in some way. In [22], we suggested two ways that would yield the same σ -model anomaly and that mimic the heterotic string: Weyl fermions in the fundamental of $SO(32)$ or a level 1 WZW model. In this paper, we focus on the fermionic formulation; in a subsequent paper we will arrive at the same conclusions by deriving the dual string Kac-Moody algebra from the fivebrane Mickelsson-Faddeev algebra [51].

The calculation of the spin 1/2 fermions on the dual string worldsheet parallels that of section 3. The fundamental representation of G_{10} can be decomposed into a sum of representations

$$fund\ G_{10} = \sum_i (l_i, c_i) \quad (5.1)$$

where l_i and c_i are irreducible representations of $G = SO(28) \times SU(2)$ and $H = SU(2)$ respectively. The number of left-hand spinor multiplets in the representation l_i of G is given by an index theorem:

$$\begin{aligned} n_i^{1/2} &= \frac{1}{4\pi^2} \int_{M_4} \left[-\frac{1}{2} tr_{c_i} F_0^2 + \frac{1}{48} dim\ c_i\ tr R_0^2 \right] \\ &= 4(dim\ c_i - 12\ r_i) \end{aligned} \quad (5.2)$$

where r_i is the ratio

$$r_i = \frac{tr_{c_i} F_0^2}{tr F_0^2} \quad (5.3)$$

This differs from (3.6) by a factor of 4 since we are going from Weyl fermions in $\tilde{d} = 6$ to Majorana–Weyl in $\tilde{d} = 2$, as opposed to Majorana–Weyl in $D = 10$ to symplectic Majorana–Weyl in $D = 6$. Since, under $SO(32) \rightarrow SO(28) \times SU(2) \times SU(2)$,

$$32 \rightarrow (28, 1, 1) + (1, 2, 2) \quad (5.4)$$

we have

$$\begin{aligned} n_{(28,1)}^{1/2} &= 4(1 - 0) = 4 = \tilde{k}_{SO(28)} \\ n_{(1,2)}^{1/2} &= 4(2 - 12) = -40 = 2\tilde{k}_{SU(2)} \end{aligned} \quad (5.5)$$

where $\tilde{k}_{SO(28)} = 4$ and $\tilde{k}_{SU(2)} = -20$ are the dual Kac–Moody levels⁸. So in analogy with (4.3) the complete sigma-model anomaly may be written⁹

$$d\tilde{H}_3 = \frac{1}{2}\tilde{\alpha}'(2\pi)^2\tilde{I}_4 \quad (5.6)$$

where from (4.1)

$$\tilde{I}_4 = \frac{1}{2(2\pi)^2}[-4\text{tr}F_{SO(28)}^2 + 40\text{tr}F_{SU(2)}^2 + \text{tr}R^2] \quad (5.7)$$

(Note that since \tilde{I}_4 is independent of the metric, there is no need for tildes on the field strengths or curvatures.) The negative sign in the $SU(2)$ case corresponds to the opposite chirality of fermions on the worldsheet. This is not yet in agreement with $2\tilde{X}_4$ of (4.11) but previous experience with the fivebrane in $D = 10$ suggests that we should not expect it to be [22]. First we note that

$$\frac{1}{2}\tilde{I}_4 - \tilde{X}_4 = \frac{1}{2(2\pi)^2}[-\text{tr}F_{SO(28)}^2 - 2\text{tr}F_{SU(2)}^2 + \text{tr}R^2] = 2X_4 \quad (5.8)$$

so that the discrepancy is twice the sigma-model anomaly of the fundamental string. Up until now we have been assuming that the supergravity \tilde{B}_2 appearing in (2.5) should be identified with the dual string \tilde{B}_2 appearing in (2.11). However, there is an ambiguity in this definition. In general, we must allow

$$\frac{1}{\tilde{\alpha}'}\tilde{B}_2(\text{supergravity}) = \frac{1}{\tilde{\alpha}'}\tilde{B}_2(\text{dual string}) + \frac{m}{\alpha'}B_2(\text{string}) \quad (5.9)$$

where, in order to preserve the quantization condition on the WZW term, m must be some integer. We find from (5.8) that the correct result is obtained by the choice¹⁰ $m = 2$. In this case,

$$\begin{aligned} d\tilde{H}_3(\text{supergravity}) &= \tilde{\alpha}'(2\pi)^2\tilde{X}_4 \\ &= \frac{\tilde{\alpha}'}{4}[-2\text{tr}F_{SO(28)}^2 + 44\text{tr}F_{SU(2)}^2 - \text{tr}R^2] \end{aligned} \quad (5.10)$$

This modification to the dual string Bianchi identity is thus a classical dual string effect (tree level in the $D = 6$ dual string expansion). This is confirmed by noting that there is no dilaton dependence and that $d\tilde{H}_3$ is independent of κ^2 and that the 2π factors cancel out. Once again, by supersymmetry, the same combination of Yang–Mills field strengths and gravitational curvatures appearing in (5.10) also appears in the dual $D = 6$ action

$$\begin{aligned} \tilde{S}^{(0)}_{YM} = \frac{1}{2\kappa^2} \int d^6x \sqrt{-\tilde{g}} e^{2\Phi} \frac{\tilde{\alpha}'}{8} \tilde{t}^{\mu\nu\rho\sigma} \Big[&-2\text{tr}F_{SO(28)\mu\nu}F_{SO(28)\rho\sigma} \\ &+44\text{tr}F_{SU(2)\mu\nu}F_{SU(2)\rho\sigma} - \text{tr}R_{\mu\nu}R_{\rho\sigma} + \dots \Big] \end{aligned} \quad (5.11)$$

⁸As far as we are aware, the numbers (4,20) appearing here have no direct connection to the numbers (4,20) describing the moduli space of $K3$.

⁹The change of sign relative to (4.3) is chosen so as to accord with the fivebrane conventions of [22] and Appendices.

¹⁰As we shall see in Appendix B this admits the $D = 10$ interpretation of $-p_1/24$ where p_1 is the Pontryagin number of $K3$ given in (3.1).

where $\tilde{t}^{\mu\nu\rho\sigma}$ is obtained from $t^{\mu\nu\rho\sigma}$ of (4.8) by replacing $g_{\mu\nu}$ with $\tilde{g}_{\mu\nu}$. Now here is the crucial step: using the duality conditions (2.3), (2.6), (2.7) and the Dirac quantization rule (2.25) with $n = 1$, this classical modification to dual string Bianchi identity (5.10) and tree level action (5.11) are seen to be identical to the string one-loop correction to the fundamental string field equations (4.14) and action (4.15), respectively. This is the main result of the paper.

6 Conclusion

According to string/string duality, the Green-Schwarz factorization of the $D = 6$ spacetime anomaly polynomial I_8 into $X_4\tilde{X}_4$ means that just as X_4 is the σ -model anomaly polynomial of the fundamental string worldsheet so \tilde{X}_4 should be the corresponding polynomial of the dual string worldsheet. To test this idea we have performed a classical dual string calculation of \tilde{X}_4 and found agreement with the quantum fundamental string result. Moreover, as we discuss in the Appendices, the same result for \tilde{X}_4 can be obtained by starting with the $D = 10$ fivebrane and using

$$\tilde{X}_4 = \int_{K3} \tilde{X}_8 \quad (6.1)$$

where \tilde{X}_8 is the anomaly polynomial of the $d = 6$ fivebrane worldvolume calculated in a previous paper [22] on string/fivebrane duality. This therefore provides an *a posteriori* justification for assumptions made in [22] on the *gravitational* contribution to \tilde{X}_8 , although we agree with [43, 44] that an *a priori* justification is still lacking. However, we disagree with these authors concerning their criticism of [22] that the gauge fermions of the covariant heterotic fivebrane cannot belong to the 32 of $SO(32)$. They claim that this is inconsistent with Strominger's result [2] that the fermions of the gauge-fixed solitonic fivebrane belong to the $(2, 28)$ of $SU(2) \times SO(28)$. The fivebrane soliton, with its $SU(2)$ instanton in the four transverse directions, breaks the spacetime plus internal symmetry $SO(1, 9) \times SO(32)$ to $SO(1, 5) \times SU(2) \times SO(28)$. (Note that the embedding $SO(1, 9) \times SO(32) \supset SO(1, 5) \times SO(29)$ [43] is not the minimal embedding and corresponds to two minimal instantons [44]). Assuming that the fermions in the unbroken phase transform as a $(16, 32)$ is entirely consistent with a $(4, 2, 28)$ in the broken phase.

Despite the success of our consistency check, however, some questions remain unresolved. Firstly, we have concentrated on the $D = 6$ string obtained from the $SO(32)$ heterotic string in $D = 10$. It would be interesting to repeat the exercise for the $E_8 \times E_8$ heterotic string and for the Type *IIB* string but we have not yet done so. Secondly, the dual string seems very different from the fundamental string. In particular, whereas for the fundamental string the number of left and right moving gauge fermions (n_L, n_R) is $(20, 4)$ and the Kac-Moody levels are $k_{SO(28)} = 1$ and $k_{SU(2)} = 1$, for the dual string we have $(60, 44)$ and $\tilde{k}_{SO(28)} = 4$ and $\tilde{k}_{SU(2)} = -20$. Although this dual heterotic string still has $n_L - n_R = 16$, there seems to be a problem with conformal invariance since we get the wrong central charge. There are some caveats to be made in this connection, however. For the purpose of calculating σ -model anomalies, we have focussed only on the massless states of the dual string worldsheet. There will also be massive Kaluza-Klein modes on the worldsheet coming from compactifying the fivebrane worldvolume on $K3$ and these could contribute to the conformal anomaly. Nor should

we forget that the fourbrane, threebrane and membrane are also present, even though they make no contribution to the σ -model anomaly. Thirdly, the $SU(2)$ Yang-Mills kinetic energy terms for the dual supergravity action (5.11) also appears to have the wrong sign, related to the wrong sign for the corresponding Kac-Moody level. However, it should be borne in mind that the fundamental action and the dual action are equivalent up to duality transformations and that $S^{(0)}_{YM}$ and $\tilde{S}^{(0)}_{YM}$ contribute to both. The difference lies only in the loop expansion: in particular what is tree-level for the fundamental string Yang-Mills kinetic energy is one-loop for the dual string ($\tilde{S}^{(0)}_{YM} = S^{(1)}_{YM}$), and vice-versa. This may be verified by converting to the appropriate σ -model metric and counting powers of e^Φ . On the subject of loop expansions, it was conjectured in [5] that the number of fivebrane loops \tilde{L}_6 was related to the Euler number of the fivebrane worldvolume $\tilde{\chi}_6$ by the formula $\tilde{\chi}_6 = 2(1 - \tilde{L}_6)$ in just the same way that the number of string loops L_2 is related to the Euler number of the string worldsheet χ_2 by the formula $\chi_2 = 2(1 - L_2)$. In the present context $\tilde{M}_6 = \tilde{M}_2 \times K3$ and the Euler number is given by $\tilde{\chi}_6 = 48(1 - \tilde{L}_2)$ and hence

$$\tilde{L}_6 = 24\tilde{L}_2 - 23 \quad (6.2)$$

In summary, the success in achieving the right anomaly has still left unanswered the question of whether the dual theory (string plus membrane, threebrane and fourbrane) is itself quantum mechanically consistent.

Perhaps the most important lesson to be learned is that the nature of the dual string depends crucially on the compactification. The inconsistencies (if there are any) may thus be a blessing in disguise: perhaps the requirement that both the fundamental *and* the dual string be consistent will thus provide a non-perturbative way of narrowing down the range of allowed superstring vacua. One might even entertain the idea that in the perfect vacuum the dual string is identical to the fundamental string. For example, requiring that the dual string be chiral with non-vanishing \tilde{X}_4 means that I_8 must be non-vanishing and hence that the spacetime theory must be chiral. This would rule out toroidal compactification, for example, which is not ruled out perturbatively¹¹. It is obviously of interest in this context to see whether four spacetime dimensions is superior to six.

7 Acknowledgements

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A Review of string/fivebrane test in $D = 10$

$D = 10$ supergravity-Yang-Mills admits two anomaly-free [57, 62] formulations: one with a three-form field strength H_3 [59, 60] and the other with a seven-form field strength \tilde{H}'_7 [61].

¹¹There is something rather asymmetrical about toroidal compactification to four spacetime dimensions: the target space duality of the fundamental string is $O(6, 22; Z)$ and that of the dual string $SL(6, Z) \times SL(2, Z)$ [3, 4, 28, 33, 42].

They are related by Poincare duality:

$$\tilde{H}'_7 = e^{-\hat{\Phi}} * H_3 \quad (\text{A.1})$$

where $\hat{\Phi}$ is the $D = 10$ dilaton and $*$ denotes the Hodge dual using the canonical metric g^c_{MN} ($M = 0, 1, \dots, 9$). Taking the exterior derivative of both sides of (A.1) reveals that the roles of field equations and Bianchi identities of the three-form version of supergravity are interchanged in going to the seven-form version. The former corresponds to the field theory limit of the heterotic string while the latter is conjectured to be the field theory limit of an extended object dual to the string: the “heterotic fivebrane” [1, 2]. Just as the 2-form potential B_{MN} couples to the $d = 2$ string worldsheet via the term

$$S_2 = \frac{1}{2\pi\alpha'} \int_{M_2} d^2\xi \frac{1}{2} \epsilon^{ij} \partial_i x^M \partial_j x^N B_{MN} = \frac{1}{2\pi\alpha'} \int_{M_2} B_2 \quad (\text{A.2})$$

where ξ^i ($i = 1, 2$) are the worldsheet coordinates and $(2\pi\alpha')^{-1}$ is the string tension, so the 6-form potential \tilde{B}_{MNPQRS} ($M = 0, 1, \dots, 9$) couples to the $\tilde{d} = 6$ fivebrane worldvolume via the term

$$\begin{aligned} \tilde{S}_6 &= \frac{1}{(2\pi)^3 \tilde{\beta}'} \int_{\tilde{M}_6} d^6 \tilde{\xi} \frac{1}{6!} \epsilon^{ijklmn} \partial_i x^M \partial_j x^N \partial_k x^P \partial_l x^Q \partial_m x^R \partial_n x^S \tilde{B}_{MNPQRS} \\ &= \frac{1}{(2\pi)^3 \tilde{\beta}'} \int_{\tilde{M}_6} \tilde{B}_6 \end{aligned} \quad (\text{A.3})$$

where $\tilde{\xi}^i$ ($i = 1, \dots, 6$) are the worldvolume coordinates and $[(2\pi)^3 \tilde{\beta}']^{-1}$ is the fivebrane tension. The two tensions obey the Dirac quantization rule [5]

$$2\kappa_{10}^2 = n(2\pi)^5 \alpha' \tilde{\beta}', \quad n = \text{integer} \quad (\text{A.4})$$

where κ_{10}^2 is the $D = 10$ gravitational constant. To within $O(\alpha')$ Chern–Simons corrections to be discussed below, H_3 is the curl of B_2

$$H_3 = dB_2 + O(\alpha') \quad (\text{A.5})$$

Similarly, up to $O(\tilde{\beta}')$ Chern–Simons corrections to be discussed below, \tilde{H}_7 is the curl of \tilde{B}_6 .

$$\tilde{H}_7 = d\tilde{B}_6 + O(\tilde{\beta}') \quad (\text{A.6})$$

We are tempted to identify \tilde{H}'_7 appearing in (A.1) with \tilde{H}_7 appearing in (A.6). However, as pointed out in [22], this identification needs to be modified when we include the gravitational Chern–Simons corrections which are of higher order in the low-energy expansion than those of Yang–Mills. Accordingly, we wrote [22]

$$\tilde{H}'_7 = \tilde{H}_7 - \frac{1}{48} \frac{\tilde{\beta}'}{\alpha'} \text{tr} R^2 H_3 \quad (\text{A.7})$$

Here $R = d\omega + \omega^2$ and the Lorentz connections $\omega = \omega_M dx^M$ are in the vector representation. The choice of coefficient $1/48$ is significant and we shall return to this later on.

In [22], the $O(\alpha')$ corrections to H_3 and the $O(\tilde{\beta}')$ corrections to \tilde{H}'_7 were examined as a test of the string/fivebrane duality conjecture. One begins with the observation of that duality mixes up string and fivebrane loops: what is a one loop effect for the string might be a tree level effect for the fivebrane, and vice versa. It is this loop mixing which allows us to test string/fivebrane duality, in spite of our ignorance of how to quantize the fivebrane. If duality is correct, we should be able to reproduce string loop effects from tree-level fivebranes!

To see this, let us first consider the well-known $SO(32)$ Yang–Mills and Lorentz Chern–Simons corrections to H_3 . Let us define $F = dA + A^2$ where the gauge fields $A = A_M dx^M$ are matrices in the fundamental representation of $SO(32)$. Let us further define

$$\begin{aligned} I_4 &= \frac{1}{(2\pi)^2} \left[-\frac{1}{2} \text{tr} F^2 + \frac{1}{2} \text{tr} R^2 \right] \\ d\omega_3 &= I_4 \\ \delta\omega_3 &= d\omega_2^1 \end{aligned} \tag{A.8}$$

then the action S_2 can be modified so as to be both gauge invariant and Lorentz provided [82]

$$\delta B_2 = \frac{1}{2} \alpha' (2\pi)^2 \omega_2^1 \tag{A.9}$$

and hence the gauge invariant field strength is given by

$$\begin{aligned} H_3 &= dB_2 - \frac{1}{2} \alpha' (2\pi)^2 \omega_3 \\ dH_3 &= -\frac{1}{2} \alpha' (2\pi)^2 I_4 \end{aligned} \tag{A.10}$$

This modification to the Bianchi identity is thus a classical string effect (i.e. tree level in the $D = 10$ string loop expansion). This is confirmed by the observation that there is no dilaton dependence and that dH_3 is independent of κ_{10}^2 and that the 2π factors cancel out.

Next we turn to the Green–Schwarz anomaly cancellation mechanism [57]. The anomaly receives contributions from the gravitino and the dimension 496 adjoint representation gauginos of the $D = 10$ theory, both of which are Majorana–Weyl. As emphasized by Green and Schwarz [57], the miracle of $SO(32)$ is that I_{12} factorizes:

$$\frac{1}{2} I_{12} = X_4 \tilde{X}_8 \tag{A.11}$$

where

$$X_4 = \frac{1}{2} I_4 \tag{A.12}$$

and

$$\tilde{X}_8 = \frac{1}{(2\pi)^4} \left[\frac{1}{24} \text{tr} F^4 - \frac{1}{192} \text{tr} F^2 \text{tr} R^2 + \frac{1}{768} (\text{tr} R^2)^2 + \frac{1}{192} \text{tr} R^4 \right] \tag{A.13}$$

The factors of $1/2$ in front of I_{12} and I_4 arise because both $D = 10$ spacetime fermions and the $d = 2$ worldsheet fermions are Majorana. Defining $\tilde{\omega}'_7$ by $\tilde{X}_8 = d\tilde{\omega}'_7$, the consistent anomaly is then cancelled by adding to the effective action

$$\Delta\Gamma_2 = -2\pi \int_{M_{10}} \left(\frac{1}{\alpha'(2\pi)^2} B_2 \tilde{X}_8 + \frac{1}{3} \omega_3 \tilde{\omega}'_7 \right) \quad (\text{A.14})$$

and recalling the transformation rule for B_2 given in (A.10). Now for B_2 normalized as in (2.1), its kinetic term is

$$\Gamma_2 = -\frac{1}{2\kappa_{10}^2} \int_{M_{10}} \frac{1}{2} e^{-\phi} H_3 \wedge *H_3 \quad (\text{A.15})$$

and hence the addition of (A.14) modifies the field equation to

$$d(e^{-\phi} * H_3) = \frac{2\kappa_{10}^2}{\alpha'(2\pi)} \tilde{X}_8 \quad (\text{A.16})$$

This modification to the field equations is thus a string one-loop effect. This is confirmed by the dilaton dependence on the left hand side and by noting that the right hand side is linear in κ_{10}^2 and involves a factor $1/(2\pi)^5$ appropriate to a one-loop Feynman integral in $D = 10$.

So far, all our considerations started with the string worldsheet. The acid test for string/fivebrane duality is to reproduce (A.16) starting from the fivebrane worldvolume. Let us define

$$\begin{aligned} \tilde{I}_8 &= \frac{1}{(2\pi)^4} \left[\frac{1}{24} \text{tr} F^4 - \frac{1}{96} \text{tr} F^2 \text{tr} R^2 + \frac{5}{768} (\text{tr} R^2)^2 + \frac{1}{192} \text{tr} R^4 \right] \\ d\tilde{\omega}_7 &= \tilde{I}_8 \\ \delta\tilde{\omega}_7 &= d\tilde{\omega}_6^1 \end{aligned} \quad (\text{A.17})$$

Then, as shown in [22] the action \tilde{S}_6 of section 2 can be modified so as to be both gauge invariant and Lorentz invariant, provided

$$\delta\tilde{B}_6 = -\tilde{\beta}'(2\pi)^4 \tilde{\omega}_6^1 \quad (\text{A.18})$$

and hence the gauge invariant field strength is given by

$$\tilde{H}_7 = d\tilde{B}_6 + \tilde{\beta}'(2\pi)^4 \tilde{\omega}_7 \quad (\text{A.19})$$

$$d\tilde{H}_7 = \tilde{\beta}'(2\pi)^4 \tilde{I}_8 \quad (\text{A.20})$$

Note that \tilde{I}_8 is not quite identical to \tilde{X}_8 but

$$\tilde{I}_8 - \tilde{X}_8 = \frac{1}{48} \frac{1}{(2\pi)^2} \text{tr} R^2 X_8 \quad (\text{A.21})$$

and so invoking (A.7), (A.10), (A.11) and (A.21), the gauge-invariant field strength \tilde{H}'_7 satisfies

$$d\tilde{H}'_7 = \tilde{\beta}'(2\pi)^4 \tilde{X}_8 \quad (\text{A.22})$$

This modification to the Bianchi identity is thus a classical fivebrane effect (tree-level in the $D = 10$ fivebrane loop expansion). This is confirmed by noting that there is no dilaton dependence and that $d\tilde{H}'_7$ is independent of κ_{10}^2 and that the factors of 2π cancel. Now here is the crucial step: using the duality condition (A.1) and the Dirac quantization rule (A.4), this classical modification to the fivebrane Bianchi identity (A.22) is seen to be identical to the string one-loop correction to the string field equations (A.16).

B $D = 6$ results from $D = 10$

In this Appendix, we reproduce the $D = 6$ dual string results of section 5 starting from the $D = 10$ dual fivebrane results of [22] reviewed in Appendix A.

First we note that the dual string worldsheet \tilde{M}_2 is obtained by compactification on $K3$ of the dual fivebrane worldvolume $\tilde{M}_6 = \tilde{M}_2 \times K3$. In particular, the 2-form of \tilde{B}_2 of (2.11) is related to the 6-form \tilde{B}_6 of (A.3) by

$$\frac{1}{2\pi\tilde{\alpha}'}\tilde{B}_2 = \frac{1}{(2\pi)^3\tilde{\beta}'}\int_{K3}\tilde{B}_6 \quad (\text{B.1})$$

where

$$\frac{1}{2\pi\tilde{\alpha}'} = \frac{1}{(2\pi)^3\tilde{\beta}'}V \quad (\text{B.2})$$

and V is the volume of $K3$. Similarly,

$$\frac{1}{\kappa_{10}^2} = \frac{V}{\kappa^2} \quad (\text{B.3})$$

So the $D = 6$ Dirac quantization rule (2.25) follows from the $D = 10$ rule (A.4). As a consequence of (B.1), we have

$$d\tilde{H}_3 = \frac{\tilde{\alpha}'}{(2\pi)^2\tilde{\beta}'}\int_{K3}d\tilde{H}_7 = \tilde{\alpha}'(2\pi)^2\int_{K3}\tilde{I}_8 \quad (\text{B.4})$$

on using (A.22). This provides us with the consistency check. Under the compactification of section 3,

$$\begin{aligned} \text{tr}F_{SO(32)}^2 &= \text{tr}F_{SO(28)}^2 + 2\text{tr}F_{SU(2)}^2 + 2\text{tr}F_0^2 \\ \text{tr}F_{SO(32)}^4 &= \text{tr}F_{SO(28)}^4 + 2\text{tr}F_{SU(2)}^4 + 6\text{tr}F_{SU(2)}^2\text{tr}F_0^2 + 2\text{tr}F_0^4 \\ \text{tr}R_{SO(1,9)}^2 &= \text{tr}R_{SO(1,5)}^2 + \text{tr}R_0^2 \\ \text{tr}R_{SO(1,9)}^4 &= \text{tr}R_{SO(1,5)}^4 + \text{tr}R_0^4 \end{aligned} \quad (\text{B.5})$$

Using the fact that $\text{tr}R_0^2 = 2\text{tr}F_0^2$, together with (3.1) we can now perform the integration bearing in mind that only F_0 and R_0 lie in the $K3$ subspace. Thus a 12th order polynomial with arbitrary coefficients a, b, c, d, e yields

$$\begin{aligned} &\frac{1}{(2\pi)^4}\frac{1}{24}\int_{K3}\left[a\text{tr}F^4 + b(\text{tr}F^2)^2 + c\text{tr}F^2\text{tr}R^2 + d(\text{tr}R^2)^2 + e\text{tr}R^4\right] \\ &= \frac{1}{(2\pi)^2}\left[4(2b+c)\text{tr}F_{SO(28)}^2 + 4(3a+4b+2c)\text{tr}F_{SU(2)}^2 + 4(c+2d)\text{tr}R_{SO(1,5)}^2\right] \end{aligned} \quad (\text{B.6})$$

In particular, from (A.13) and (A.17)

$$\begin{aligned} \int_{K3}\tilde{I}_8 &= \frac{1}{2}\tilde{I}_4 \\ \int_{K3}\tilde{X}_8 &= \tilde{X}_4 \end{aligned} \quad (\text{B.7})$$

in complete agreement with (5.6) and (5.10).

It is now of interest to see how the $D = 6$ shift (5.9) follows from the $D = 10$ shift (A.7). In particular, the factor $1/48$ was rather mysterious from $D = 10$ point of view [22]. Now we see the significance of 48 in the $D = 6$ context as the Pontryagin number of $K3$, and the integer m is given by

$$m = -\frac{1}{48(2\pi)^2} \int_{K3} tr R_0^2 = -\frac{p_1}{24} = 2 \quad (B.8)$$

The above results (B.6), (B.7) and (B.8) thus provide an *a posteriori* justification for the choice in [22] of polynomial \tilde{I}_8 in (A.17) and shift in (A.21). The point being that the coefficients c, d, e chosen in [22] on the basis of an educated guess of the mixed and gravitational fivebrane sigma-model anomalies, yield the well-known gravitational sigma-model anomaly polynomial for the string as well as the correct Yang-Mills terms.

In a similar way we may derive the terms in the $D = 6$ dual action (5.11) quadratic in the Yang-Mills field strengths and gravitational curvature. We begin with the tree-level heterotic fivebrane action which, as discussed in [7], must be *quartic* in the field strengths:

$$\begin{aligned} \tilde{S}^{(0)}_{YM} = & \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-\tilde{g}} e^{2\tilde{\Phi}/3} \frac{\tilde{\beta}'}{24} \tilde{t}^{IJKLMNPQ} [\text{tr} F_{IJ} F_{KL} F_{MN} F_{PQ} \\ & - \frac{1}{8} \text{tr} F_{IJ} F_{KL} \text{tr} R_{MN} R_{PQ} + \frac{1}{32} \text{tr} R_{IJ} R_{KL} \text{tr} R_{MN} R_{PQ} + \frac{1}{8} \text{tr} R_{IJ} R_{KL} R_{MN} R_{PQ} + \dots] \end{aligned} \quad (B.9)$$

where $\tilde{g}_{MN} = e^{-\Phi/6} g^c_{MN} = e^{-2\Phi/3} g_{MN}$ is the fivebrane σ -model metric [5] and where

$$\begin{aligned} \tilde{t}^{IJKLMNPQ} = & -\frac{1}{2} (g^{IK} g^{JL} - g^{IL} g^{JK}) (g^{MP} g^{NQ} - g^{MQ} g^{NP}) \\ & - \frac{1}{2} (g^{KM} g^{LN} - g^{KN} g^{LM}) (g^{PI} g^{QJ} - g^{PJ} g^{QI}) \\ & - \frac{1}{2} (g^{IM} g^{JN} - g^{IN} g^{JM}) (g^{KP} g^{LQ} - g^{KQ} g^{LP}) \\ & + \frac{1}{2} (g^{JK} g^{LM} g^{PN} g^{QI} + g^{JM} g^{NK} g^{LP} g^{QI} + g^{JM} g^{NP} g^{KQ} g^{LI} + \text{permutations}) \end{aligned} \quad (B.10)$$

Integrating over $K3$ as in (B.6) above yields the $D = 6$ dual string action (5.11).

Finally, we note that the fundamental string tree-level but 4 worldsheet loop contribution to the effective action in $D = 10$ takes the form [84]

$$\begin{aligned} & \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \alpha' t^{IJKLMNPQ} [b \text{tr} F_{IJ} F_{KL} \text{tr} F_{MN} F_{PQ} \\ & - 2b \text{tr} F_{IJ} F_{KL} \text{tr} R_{MN} R_{PQ} + b \text{tr} R_{IJ} R_{KL} \text{tr} R_{MN} R_{PQ} + e \text{tr} R_{IJ} R_{KL} R_{MN} R_{PQ}] \end{aligned} \quad (B.11)$$

and hence integrating over $K3$ as in (B.6) above yields a vanishing contribution to the $D = 6$ string action (4.7) and so does not interfere with any of our previous results.

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